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On Alternation Numbers in Nonlinear Chebyshev Approximation

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1. INTRODUCTION

The aim of this paper is to enlarge the scope of the principle of alternation in Chebyshev approximation by developing alternation criteria for uniqueness, strong uniqueness, and continuous dependence of locally and globally best approximations. The underlying numbers of alternation points depend only on the approximating functions, this way describing some of their basic approximation properties. Parameters are eliminated, but the signs of alternants are taken into account in order to cover the approximation by positive exponential sums and analogously defined special γ-polynomials (Braess [4, 7, 8] and Schmidt [16, 17]). Parameter-free generalizations of varisolvency and normality, based only on alternation, are included. Our examples lead to new theorems on strong uniqueness and continuity of the Chebyshev operator and put a series of well-known results into the framework of this paper.

2. PRELIMINARIES

Let X be a space of real-valued functions on a totally ordered set Q such that the Chebyshev norm

$$\| x \| := \sup_{t \in Q} | x(t)|$$

is finite for every $x \in X$. For simplicity, a weight function is suppressed. For elements v of a family V of functions in X approximating a function $x \in X$ the notations

$$\begin{aligned}
\mathrm{BA}(x, V) &:= \{v \in V \mid \| x - v \| \leqslant \| x - w \| \text{ for all } w \in V\},\\
\mathrm{LBA}(x, V) &:= \{v \in V \mid \text{there is a neighborhood } U \text{ of } v \text{ with } v \in \mathrm{BA}(x, U \cap V)\},\\
P(x, v, V) &:= \{w \in X \mid v - w \in V, \| x - v + w \| < \| x - v \|\},\\
\bar{P}(x, v, V) &:= \{w \in X \mid v - w \in V, \| x - v + w \| \leqslant \| x - v \|, w \neq 0\},\\
I &:= \{+1, -1\}
\end{aligned}$$

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for the sets of globally and locally best approximations and feasible perturbations are introduced. The symbol S will be used for nonvoid subsets of I.

For $x \in X$, $R \subset Q$, and $S \subset I$ the symbol $\mathrm{Alt}(x, R, S)$ (resp. $\mathrm{alt}(x, R, S)$, $\overline{\mathrm{alt}}(x, R, S)$) denotes the supremum of all integers k such that there are points $t_1 < t_2 < \cdots < t_k$ in R and a sign $s \in S$ satisfying

$$x(t_i)(-1)^{i-1}s = \|x\| \qquad (1 \leqslant i \leqslant k)$$
$$\text{resp. } x(t_i)(-1)^{i-1}s > 0, \qquad x(t_i)(-1)^{i-1}s \geqslant 0 \qquad \text{for } i = 1,\ldots, k.$$

For subsets Y of X the notation

$$\sup \mathrm{Alt}\,(Y, R, S) := \sup_{y \in Y} \mathrm{Alt}\,(y, R, S)$$

is introduced, where sup resp. Alt can be replaced by inf resp. alt or $\overline{\mathrm{alt}}$.

The usual maximal number of alternation points of x on Q is $\mathrm{Alt}(x, Q, I)$. In addition, the sign of x in the first alternation point may be prescribed by choosing $S = \{+1\}$ or $\{-1\}$. This is important for the approximation by positive exponential sums and γ-polynomials [4, 7, 8, 16, 17]. The restriction to subsets R of Q is necessary to cover results of spline approximation (Schumaker [18, 19], Braess [5], and Arndt [1]), as can be seen from Examples 5–7 below.

3. Generalized Alternation Conditions for Best Approximations

The criteria for strong uniqueness given in the next section are designed to fit into the framework of alternation criteria in Chebyshev approximation. The hierarchy of conditions is described (in generalized form) in Fig. 1.

Following Meinardus and Schwedt [13], a "tangent set" $C \subset X$ depending on $v \in V$ and V is used in Fig. 1, which has to satisfy the condition

$$O \in \mathrm{BA}(x - v, C) \qquad \text{for all } x \in X \text{ with } v \in \mathrm{LBA}(x, V). \tag{1}$$

The set of sets $C \subset X$ with (1) is denoted by $T(v, V)$. There are several intrinsic definitions of tangent spaces or cones in the literature (see, e.g., [6, 8, 11, 13, 15, 23, 24]). Those known to the author all satisfy (1) and are contained in the cone introduced by Dubovickii and Miljutin [11]. The necessary condition for best approximations is then implied by defining

$$N(v, V, R, S) := \inf_{\substack{v \in \mathrm{BA}(x, V)\\ x \in X}} \mathrm{Alt}\,(v - x, R, S),$$
$$n(R, C, S) := N(O, C, R, S)$$

for $v \in V \subset X$, $R \subset Q$, $S \subset I$, and $C \in T(v, V)$.

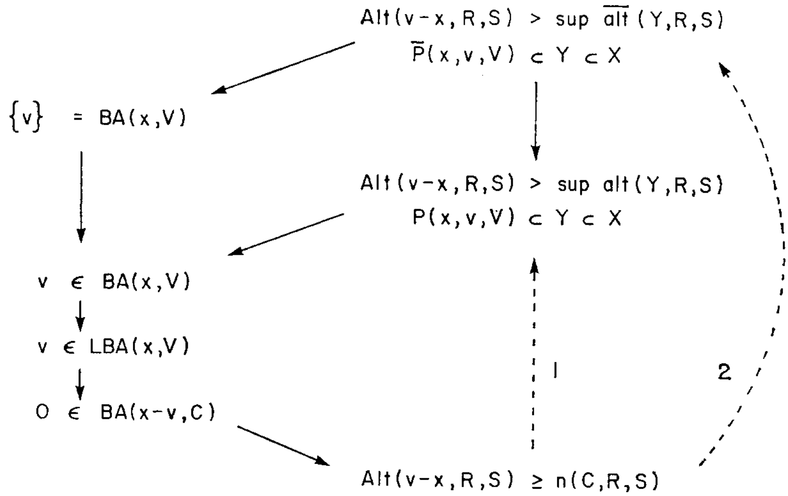

FIG. 1. $v \in V \subset X$, $x \in X$, $R \subset Q$, $S \subset I$, $C \in T(v, V)$. Arrows indicate implications (dotted when holding under additional assumptions).

Of course, the calculation of necessary numbers of alternation points is only reduced to the corresponding problem for the tangent set; the latter usually is a problem of analytical nature and far more complicated, as the following theorem shows, which is a general application of the "perturbation techniques" in Chebyshev approximation [20].

THEOREM 1. *For $Q := [a, b] \subset \mathbb{R}$ and $v \in V \subset X \subset C(Q)$ let $m(v, V)$ be the supremum of zero and all integers j such that for all k, $1 \leqslant k \leqslant j$, all partitions of Q into k subintervals by points $a = r_0 < \cdots < r_k = b$, all signs $s \in I$ and all constants $\delta > 0$ there is a $w_\delta \in X$ with $\| w_\delta \| < \delta$ and $w_\delta + v \in V$ satisfying $w_\delta(t)\, s < 0$ in $[a, b]$ for $k = 1$ and*

$$w_\delta(t)(-1)^{i-1} s > 0 \qquad \text{for } t \in [r_i + \rho_\delta, r_{i+1} - \rho_\delta],\ 1 \leqslant i \leqslant k-2,$$
$$t \in [a, r_1 - \rho_\delta],\ i = 0,\ t \in [r_{k-1} + \rho_\delta, b],\ i = k-1 \tag{2}$$

with $\rho_\delta > 0$, $\rho_\delta \to 0$ for $\delta \to 0$. Then $N(v, V, Q, I) = m(v, V) + 1$.

Proof. For $x \in X \setminus V$ and $k := \mathrm{Alt}(v - x, Q, I)$ one has to show that $k \geqslant m(v, V) + 1$. Now let $t_1 < \cdots < t_k$ be chosen maximally with $1 < k < \infty$ and $(v - x)(t_i)(-1)^{i-1} s = \| v - x \|$, $s \in I$. The case $k = 1$ is obvious. There are points $r_i \in (t_i, t_{i+1})$, $1 \leqslant i \leqslant k - 1$, and $\epsilon > 0$ such that

$$-\| v - x \| + \epsilon \leqslant (v - x)(t)(-1)^i s \leqslant \| v - x \| \tag{3}$$

for $t \in [r_i, t_{i+1}]$, $1 \leqslant i \leqslant k-2$, and $t \in [r_{k-1}, b]$, $i = k-1$, and

$$-\| v - x \| \leqslant (v - x)(t)(-1)^i s \leqslant \| v - x \| + \epsilon \tag{4}$$

for $t \in [t_i, r_i]$, $2 \leqslant i \leqslant k-1$, and $t \in [a, r_1]$, $i = 1$, hold. This is easily achieved by any choice of r_i in the interval

$$Q_i := \left(t_i, \min \left\{ t \,\middle|\, \begin{matrix} (v-x)'(t)(-1)^i s = \| v - x \| \\ t \in (t_i, t_{i+1}] \end{matrix} \right\}\right)$$

and a sufficiently small $\epsilon > 0$. In addition, one can have these estimates with a fixed ϵ uniformly for all r_i varying in a compact subinterval Q_i' of Q_i. If $k \leqslant m(v, V)$, then for every positive $\delta < \epsilon$ there is a function w_δ with the properties stated above. Since the estimates (3) and (4) may be assumed to hold uniformly in small neighborhoods of the r_i, they hold for the $r_i \pm \rho_\delta$ replacing r_i when δ is sufficiently small. From (2), (3), (4), and $\| w_\delta \| < \delta < \epsilon$ it follows that $v + w_\delta \in V$ is a better approximation to x than v. Therefore $k \geqslant m(v, V) + 1$.

The dotted arrows 1 and 2 in Fig. 1 indicate implications holding under the conditions

$$\sup \operatorname{alt} (Y, R, S) < n(C, R, S), \tag{5}$$

$$\sup \overline{\operatorname{alt}} (Y, R, S) < n(C, R, S). \tag{6}$$

These provide a parameter-free generalization of results of approximation by varisolvent families (Rice [15]). The assumptions (6) and the weakened form (5) may be viewed as generalized parameter-free varisolvency conditions.

4. Criteria for Strong Uniqueness

The concept of strong (local) uniqueness is important for tangential characterizations [23, 24], global analysis [6, 8, 9], and the Lipschitz continuity of the Chebyshev operator (Theorem of Freud, see, e.g., [10]). Therefore it is convenient to have simple criteria for (local or global) strong uniqueness.

Definition. (a) $v \in V \subset X$ is a strongly unique (locally) best approximation to $x \in X$ iff there is a $K > 0$ such that

$$\| x - w \| \geqslant \| x - v \| + K \cdot \| v - w \|$$

holds for all $w \in V$ (for all $w \in V \cap W$, where W is a neighborhood of v). This is abbreviated by $v \in \text{SUBA}(x, V)$ ($v \in \text{SULBA}(x, V)$).

(b) Let $v \in V \subset X$, $R \subset Q$, and $S \subset I$. The symbol $\operatorname{Lim} \overline{\operatorname{alt}}(v, V, R, S)$ denotes the supremum of all integers k such that there are k points $t_1 < \cdots < t_k$ in R, a sign $s \in S$, and a bounded sequence $\{v_i\} \subset V \backslash \{v\}$ such that the limits

$$\lim_{i \to \infty} \frac{v - v_i}{\| v - v_i \|} (t_j)(-1)^{j-1} s \qquad (1 \leqslant j \leqslant k) \tag{7}$$

exist and are nonnegative.

(c) A constant $\lim \overline{\operatorname{alt}}(v, V, R, S)$ depending only on the local structure of V near v is defined analogously by restricting the above definition to sequences $\{v_i\} \subset V \backslash \{v\}$ uniformly converging to v.

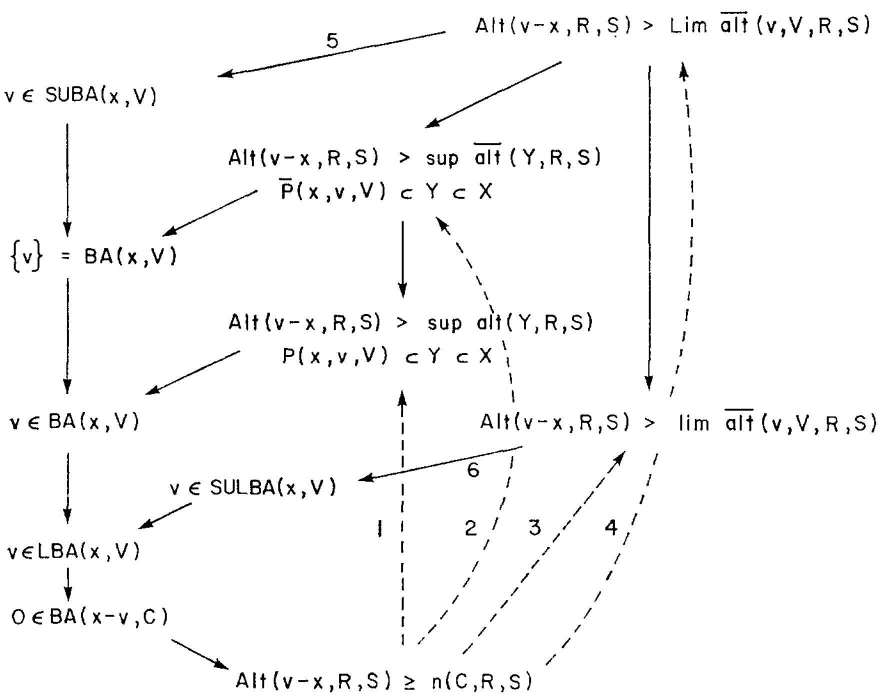

FIG. 2. See caption of Fig. 1.

Proof for the implications indicated by arrows 5 and 6 in Fig. 2.

Assume to the contrary that there are sequences $\{v_i\} \subset V$ and $\{k_i\} \to 0$ such that $k_i > 0$ and

$$\| v_i - x \| < \| v - x \| + k_i \| v - v_i \| \qquad (i = 1, 2, \ldots). \tag{8}$$

Then $v_i \neq v$, and the boundedness of $\{v_i\}$ follows from

$$\| v_i - x \| < \frac{1 + k_i}{1 - k_i} \| v - x \| .$$

If Lim is replaced by lim, the sequence $\{v_i\}$ can be assumed to be convergent to v. In both cases (8) yields for $k > \operatorname{Lim} \overline{\operatorname{alt}}\,(v, V, R, S)$ (resp. $\lim \overline{\operatorname{alt}}\,(v, V, R, S)$) points $t_1 < \cdots < t_k$ in R and a sign $s \in S$ with $(v - x)(t_j)(-1)^{j-1} s = \| v - x \|$, $1 \leqslant j \leqslant k$, the inequalities

$$\frac{v_i - v}{\| v_i - v \|}(t_j)(-1)^{j-1} s < k_i \qquad (1 \leqslant j \leqslant k).$$

By passing to a subsequence the nonnegativity of (7) on k points follows, which is a contradiction. The implications indicated by the dotted arrows 3 and 4 hold whenever the conditions

$$\lim \overline{\operatorname{alt}}\,(v, V, R, S) < n(C, R, S), \tag{9}$$

$$\operatorname{Lim} \overline{\operatorname{alt}}\,(v, V, R, S) \leqslant \sup \overline{\operatorname{alt}}\,(Y, R, S) \tag{10}$$

are satisfied.

Inequality (10) may be viewed as a generalized "normality" condition, because it implies continuity of the Chebyshev operator when combined with general varisolvency in the form of (6). Furthermore, (14) is a sufficient condition for the equivalence of strongly unique best approximations and critical points, cf. the examples of this equivalence in [6, 8] (see Examples 3 and 4 below).

5. Applications

In this section the constants appearing in Figs. 1 and 2 are calculated for a series of examples in order to obtain special cases of the above alternation criteria. For simplicity, $Q := [0, 1]$ is assumed and the notation $V_v := \{v - u \mid u \in V, u \neq v\}$ is introduced.

Example 1. If V is an n-dimensional Haar subspace of $X := C(Q)$, then Theorem 1 implies $N(v, V, Q, I) \geqslant m(v, V) + 1 = n + 1$ for $v \in V$, since there are functions in V with k sign changes, $0 \leqslant k \leqslant n - 1$, arbitrarily near k prescribed points in $(0, 1)$. On the other hand, for $Y = V_v$, $x \in X$, $R \subset Q$, $S \subset I$, and $C = V$ all other constants in Fig. 2 except for $\operatorname{Alt}(v - x, R, S)$ are equal to n. Therefore all statements in Fig. 2 are equivalent.

Example 2. If V is a family satisfying the local and global Haar condition [13] with degree n at $v \in V \subset X = C(Q)$, then

$$\sup \overline{\operatorname{alt}}\,(V_v, Q, I) \leqslant n < n + 1 = n(C, Q, I) \tag{11}$$

holds, where C stands for the (Haar) tangent space at v. Of course, the left and right parts of (11) resemble the global and local Haar conditions. Since (6) holds, all statements in Fig. 1 are equivalent for $Y = V_v$, $R = Q, S = I$.

The proofs of Barrar and Loeb in [2] can be modified to establish

$$\operatorname{Lim} \overline{\operatorname{alt}} (v, V, Q, I) \leqslant \max (n, \lim \overline{\operatorname{alt}} (v, V, Q, I))$$
$$\leqslant \max (n, \sup \overline{\operatorname{alt}} (C, Q, I)) \leqslant n$$

stepwise for elements $v \in V$ having maximal degree h, yielding strong uniqueness by (10) and Fig. 2.

EXAMPLE 3. If $M \subset C(Q)$ is a C^1-manifold with boundaries in the sense of [6], then for $v \in M$ and the cone C_vM in [6] the inequality $\lim \overline{\operatorname{alt}} (v, M, R, S) \leqslant \lim \overline{\operatorname{alt}} (O, C_vM, R, S)$ holds for all $R \subset Q$, $S \subset I$. This is easily proved by mapping (7) into C_vM by [6, (3.1)]. By generalizing the concept of "Haar embedding" to the condition $\lim \overline{\operatorname{alt}} (O, V_vM, R, S) < N(O, C_vM, R, S)$ on the tangent cone C_vM for a special choice of R and S, one gets a characterization of critical points and (strongly unique) local best approximations via (9) and Fig. 2. This is a generalization of [6, Satz 7.1].

EXAMPLE 4. For the family E_n of (generalized) exponential sums

$$v(x) = \sum_{i=1}^{l} P_i(x) \exp (r_i x) \tag{12}$$

with real frequencies $r_i < \cdots < r_l$ and real polynomials $P_i(x)$ of degree p_i fulfilling

$$p_1 + \cdots + p_l + l =: k(v) =: k \leqslant n$$

one gets the estimates

$$\sup \overline{\operatorname{alt}} ((E_n)_v, Q, I) \leqslant n + k,$$
$$n(C, Q, I) > n +$$

where C denotes the tangent space obtained by differentiation of (12) with respect to the parameters (see, e.g., Rice [15] and Braess [4, 7]). In case $k = l$ (i.e., for noncoalescing frequencies) one has the situation of Example 2.

The differential equation arguments introduced by Werner [21] and Schmidt [16] easily show for elements v of E_n having maximal degree $k(v) = n$ (with $l < n$ admitted) that the uniform convergence of a bounded sequence $\{v_i\}$ of E_n to v implies the uniform convergence of a subsequence of $(v_i - v)/\| v_i - v \|$ to an element w of E_{2n}. The construction of a tangent cone $W = W(v)$ by Braess [8] shows that $w \in W$. Consequently,

$$\lim \overline{\operatorname{alt}} (v, E_n, Q, S) \leqslant \sup \overline{\operatorname{alt}} (W_0, Q, S) \tag{13}$$

for any $S \subset I$. For a suitable choice of $S \subset I$ and a parameter $L = L(v)$ between l and $k = n$ the arguments of Braess establish

$$\sup \overline{\mathrm{alt}}\,(W_0, Q, S) \leqslant n + L - 1 < n + L \leqslant n(W, Q, S).$$

This can be viewed as a generalized varisolvency condition (6) for the cone W. Combined with (13) this inequality implies (9) and local strong uniqueness of locally best approximations, putting results of Braess ([8, Section 12], for the general case of normal families of γ-polynomials with extended totally positive kernels of order $2n$) into the framework of this paper.

EXAMPLE 5. For the family $E_n{}^+$ of termwise positive exponential sums

$$v(x) := \sum_{i=1}^{k} q_i \exp(r_i x), \qquad k = k(v) \leqslant n,\ q_i > 0,\ r_i \in \mathbb{R},$$

one has [4]

$$\sup \mathrm{alt}\,((E_n{}^+)_v, Q, S_k) \leqslant 2k < 2k + 1 \leqslant n(C, Q, S_k).$$

Here $S_n = I$ and $S_k = \{+1\}$ if $k < n$, and $C = C(v)$ stands for the cone of exponential sums

$$\sum_{i=1}^{k} (a_i x + c_i) \exp(r_i x) + \sum_{j=k+1}^{n} d_j \exp(s_j x),$$

$$a_i, c_i, d_j, s_j \in \mathbb{R},\ 1 \leqslant i \leqslant k,\ k + 1 \leqslant j \leqslant n,\ d_j \geqslant 0.$$

This illustrates the use of prescribing signs by choosing a suitable $S \subset I$. Using the same arguments as in Examples 2 and 4 one gets $\mathrm{Lim}\,\overline{\mathrm{alt}}\,(v, E_n{}^+, Q, I) \leqslant 2n < 2n + 1 \leqslant n(C, Q, I)$ for nondegenerate elements v of $E_n{}^+$ (with $k = n$). Thus (10) holds and Fig. 2 implies strong uniqueness.

The results of Schmidt [17] suggest that strong uniqueness of a best approximation $v \in E_n{}^+$ to a function $x \in C(Q)$ could hold as well (but possibly only locally) in the cases:

(1) $k < n$, $v - x$ alternates in exactly $2k + 1$ points;

(2) $k < n$, $v - x$ attains $\| v - x \|$ at 0 and 1.

By enlarging the considered interval and adding degenerating terms like $v_j(x) := \exp(-j(1 + x + \epsilon))$ to $\exp(x)$ in the example of Schmidt [17] one concludes that (1) is not sufficient and (2) is necessary for local strong uniqueness in case $k < n$.

The sufficiency of (2) is a consequence of

THEOREM 2. *Let* $v \in E_k^+$, $k < n$, *and*

$$R \in \{\{t_1, \ldots, t_{2m+1}\} \mid k \leqslant m, 0 = t_1 < \cdots < t_{2m+1} = 1\}$$

be given. Then $\operatorname{Lim} \overline{\operatorname{alt}}\, (v, E_n^+, R, \{+1\}) \leqslant 2k$.

Proof. Assume to the contrary that for a bounded sequence $\{v_i\} \subset E_n^+$ and a sequence $\{k_i\} \to \mathbb{R}$ tending to zero the inequalities

$$\frac{v - v_i}{\| v - v_i \|}(t_j)(-1)^{j-1} \geqslant -k_i \qquad (1 \leqslant j \leqslant 2k + 1, i = 1, 2, \ldots) \qquad (14)$$

hold, where $t_1 = 0$, $t_{2k+1} = 1$, and $2(m - k)$ interior points of R are dropped. Passing to subsequences is allowed and will not be noticed in the sequel.

Assertion 1. It can be assumed that the frequencies of v_i are uniformly bounded and $\| v_i - v \| \to 0$ for $i \to \infty$.

To prove this, let v_i^2 be the sum of those terms of v_i having unbounded frequencies for $i \to \infty$, and $v_i^1 := v - v_i^2$. Then the v_i^2 and each of their terms are uniformly bounded, thus converging to zero in (0, 1). This implies via (14) the inequalities

$$\lim_{i \to \infty} (v_i^1 - v)(t_j)(-1)^{j-1} \leqslant 0 \qquad (1 \leqslant j \leqslant 2k + 1).$$

By counting the zeros of the limit function of $v_i^1 - v$ and checking signs at the boundary one has $\| v_i^1 - v \| \to 0$. Since each term of v_i^2 is bounded by the maximum of its boundary values, which by (14) are bounded by $k_i \cdot \| v - v_i \| + \| v - v_i^1 \|$, one can bound $v_i^2(t)$ by

$$0 \leqslant v_i^2(t) \leqslant n \cdot (k_i \| v - v_i \| + \| v - v_i^1 \|) \exp(-dz_i). \qquad (15)$$

Here z_i denotes the smallest absolute value of the frequencies of v_i^2 and t has a minimum distance d from the boundary. Setting $d := \max(t_1, 1 - t_{2k})$ and $\epsilon_i := n \exp(-dz_i)$, and using again the information on the signs at the boundary, one can use (14) and (15) to get

$$(v_i^1 - v)(t_j)(-1)^{j-1} \leqslant k_i \| v_i - v \| + \epsilon_i \cdot (k_i \| v - v_i \| + \| v - v_i^1 \|) \qquad (16)$$

for $j = 1, \ldots, 2k + 1$.

Another consequence of (15) is

$$\| v_i - v \| \leqslant \frac{1 + n}{1 - nk_i} \| v_i^1 - v \|,$$

which combined with (16) implies (14) for v_i^1 instead of v_i and $k_i + o(1)$ instead of k_i, proving Assertion 1.

Assertion 2. The limit function w of $(v_i - v)/\| v_i - v \|$ belongs to the cone of functions

$$\sum_{i=1}^{k} (a_i x^2 + b_i x + c_i) \exp(r_i x) + \sum_{j=k+1}^{n} d_j \exp(s_j x),$$

$$a_i, d_j \geqslant 0, b_i, c_i, s_j \in \mathbb{R}, s_j \neq r_i, 1 \leqslant i < k+1 \leqslant j \leqslant n,$$

where $r_1 < \cdots < r_k$ are the frequencies of

$$v(x) =: \sum_{j=1}^{k} q_j \exp(r_j x), \qquad q_j > 0, 1 \leqslant j \leqslant k < n.$$

Proof. By differential equation arguments qne can easily show that for each frequency r_j of v there are (possibly several) terms of v_i with frequencies converging to r_j. Let these terms be collected into v_{ij}. With $w_j(x) := q_j \exp(r_j x)$ the identity

$$\frac{v_i - v_{ij} - v + w_j}{\| v_{ij} - w_j \|} = \frac{v_i - v}{\| v_i - v \|} \cdot \frac{\| v_i - v \|}{\| v_{ji} - w_j \|} - \frac{v_{ij} - w_j}{\| v_{ij} - w_j \|} \tag{17}$$

easily leads to the boundedness of the quotients $\| v_{ij} - w_j \|/\| v_i - v \|$ and to $\| v_{ij} - w_j \| \to 0$ for $i \to \infty$. This implies that Assertion 2 can be proved termwise; it suffices to show that (using new notations) the functions

$$v_i(x) := \sum_{j=1}^{l} q_{ij} \exp(r_{ij} x), \qquad q_{ij} > 0, r_{ij} \to r,$$

$$v(x) := q \exp(rx), \qquad q > 0,$$

with

$$\epsilon_i := \| v_i - v \| \to 0$$

satisfy

$$\lim_{i \to \infty} (v_i - v)(x)/\epsilon_i = (ax^2 + bx + c) \exp(rx)$$

uniformly in x for $a \geqslant 0$, $b, c \in \mathbb{R}$. This follows from the uniform convergence of (a subsequence of) $(v_i - v)/\epsilon_i$ and its derivatives and from the decomposition

$$(v_i - v)(x) = (v_i - v)(0) \exp(rx) + x \exp(rx) \left[\left(\frac{d}{dx} - r \right) \Big|_{x=0} (v_i - v) \right]$$

$$+ \sum_{r=1}^{l} q_{ij}(r_{ij} - r)^2 \Delta_t^2(r, r, r_{ij}) \exp(tx),$$

using generalized divided differences in the notations of [22].

EXAMPLE 6. Let $S_{n,k}(y_1, \ldots, y_k)$ denote the linear space of Chebyshevian spline functions of degree n with (possibly multiple) fixed knots $y_1, \ldots, y_k$ in $[0, 1] =: Q$ (in the notation of [5]). One gets for $s \in S_{n,k}(y_1, \ldots, y_k)$ in each interval $Q_{p,q} := [y_p, y_{p+q+1}]$, $1 \leqslant p < p + q + 1 \leqslant k$ the inequalities

$$\sup \operatorname{alt}\, ((S_{n,k}(y_1, \ldots, y_k))_s, Q_{p,q}, I) \leqslant n + q + 1$$

and

$$\operatorname{Alt}\, (s - x, Q_{r,t}, I) \geqslant n + t + 1$$

for some interval $Q_{r,t}$, $1 \leqslant r < r + t + 1 \leqslant k$, when s is a best approximation to $x \in X := C(Q)$ with respect to $S_{n,k}(y_1, \ldots, y_k)$. The implications in Fig. 1 can then be verified for any fixed $x \in X$. But the framework of Section 3 of this paper does not fit this situation because of the dependence of the intervals $Q_{r,t}$ on the approximated function x.

The arguments of Schumaker [18, proof of Theorem 3.1] can easily be used to get the following criterion for strongly unique best approximations:

THEOREM 3. *The estimation* $\operatorname{Lim} \overline{\operatorname{alt}}\, (s, S_{n,k}(y_1, \ldots, y_k), R, I) \leqslant n + k + 1$ *holds for every* $s \in S_{n,k}(y_1, \ldots, y_k)$ *and any set* R *from the set*

$$\left\{ \{t_1, \ldots, t_{n+k+2}\} \,\middle|\, \begin{matrix} t_1 \leqslant \cdots \leqslant t_{n+k+2} \\ t_{i+1} < y_i < t_{n+i+1}\ 1 \leqslant i \leqslant k \end{matrix} \right\}, \tag{18}$$

implying via Fig. 2 *that* $s \in S_{n,k}(y_1, \ldots, y_k)$ *is a strongly unique best approximation to* $x \in X$, *when a set* R *of alternation points belonging to* (18) *exists.*

EXAMPLE 7. For the set $S_{n,k}$ of Chebyshevian spline functions of degree n with k free (possibly multiple) knots in $Q = [0, 1]$ one has

$$\sup \operatorname{alt}\, ((S_{n,k})_s, Q_{p,q}, I) \leqslant n + k + l + 1$$

for any $s \in S_{n,k}$ with distinct knots $0 \leqslant x_0 < \cdots < x_{r+1} \leqslant 1$ and degree l in $Q_{p,q} := [x_p, x_{p+q+1}]$.

As in Example 6 there is no appropriate formulation of a necessary condition for best approximations, which does not involve the approximated function.

But the methods of Section 4 can be applied again to give a criterion for strongly unique best approximations, which sharpens a uniqueness theorem of Arndt [1]:

THEOREM 4. *The estimate*

$$\operatorname{Lim} \overline{\operatorname{alt}}\, (s, S_{n,k}, R, I) \leqslant n + 2k + 1 \tag{19}$$

holds for every $s \in S_{n,k} \backslash S_{n,k-1}$ *possessing the knots* $y_1 \leqslant \cdots \leqslant y_k$ *and any set* R *from the set*

$$\left\{ \{t_1, \ldots, t_{n+2k+2}\} \;\middle|\; \begin{array}{l} t_1 < \cdots < t_{n+2k+2} \\ t_{2i+1} < y_i < t_{n+2i} \; 1 \leqslant i \leqslant k \end{array} \right\} \tag{20}$$

implying via Fig. 2 *that* s *is a strongly unique best approximation to* $x \in X = C(Q)$, *if a set of alternation points belonging to* (20) *exists.*

Proof. To cope with the possible noncontinuity of limits of $(s_i - s)/\| s_i - s \|$ for $s_i \in S_{n,k} \backslash \{s\}$ one has to replace Lemmas 2.3, 2.2, and 3.1 of [1] by the statements

(a) (i) If $s \in S_{n,k}$ having knots $x_1 \leqslant \cdots \leqslant x_k$ satisfies $s(t_i)(-1)^i \sigma \geqslant 0$, $1 \leqslant i \leqslant n + k + 1$, with $\sigma \in I$ for points $t_1 < \cdots < t_{n+k+1}$, then s vanishes between some of the t_i or $t_i < x_i < t_{n+i+1}$ holds for $i = 1, \ldots, k$.

(ii) If s alternates in $n + k + 2$ points, then s vanishes between two of them.

(b) Let the assumptions in (a) (i) prevail; If, in addition, one has $t_{i+1} < x_i < t_{n+i+1}$ for $i = 1, \ldots, k$, then $s \equiv 0$ on Q.

(c) If $s \in S_{n,k} \backslash S_{n,k-1}$ has the knots $x_1 \leqslant \cdots \leqslant x_k$, then for every bounded sequence $\{s_i\} \subset S_{n,k} \backslash \{s\}$ there is no set of points $t_1 < \cdots < t_{n+2k+2}$ with $t_{2i+1} < x_i < t_{n+2i}$ for $i = 1, \ldots, k$ and

$$\lim_{j \to \infty} \frac{s_j - s}{\| s_j - s \|} (t_i)(-1)^i \sigma \geqslant 0 \qquad \text{for } \sigma \in I, 1 \leqslant i \leqslant n + 2k + 2.$$

Using only (a) one can easily prove Theorem 3 (in case of a noncontinuous s). Statement (b) can be reduced to [1, Lemma 2.2] by using an argument of Schumaker [19] concerning perturbations of the knots at jump locations. By slight modifications of Arndt's proofs for the corresponding lemmas, one gets (a) and (c). Finally, Statement (c) proves the theorem. The above arguments again indicate that the consideration of alternation points instead of zeros is appropriate.

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